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Writing Mathematical Dialogues as a Professional Advancement Tool for Mathematics Teachers

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Abstract **– In-service teachers are often surprised to find themselves at a loss for words in the mathematics classroom. This feeling is not limited to the first day of class or to beginning teachers. Even experienced teachers describe unexpected classroom situations in which they cannot find the proper words to respond or to explain or mediate ideas. The teaching routine is fraught with on time decisions teachers must make. The current research attempts to train and prepare teachers to use written fictional dialogues in managing spoken mathematical classroom discourse as part of their decision-making process in unexpected classroom situations. The research shares Schoenfeld's assumption regarding teachers' decision - making. "People's in-the-moment decision making is a function of their knowledge and resources, goals, and beliefs and orientations. Their decisions and actions can be 'captured' (explained and modeled) in detail using only these constructs" [19].**

Keywords **– Decision Making in the Mathematics Classroom, Hypothetical Learning Trajectory, In-service Teachers Pre-service Teachers, Professional Development, Writing Mathematical Dialogues.**

I. INTRODUCTION

Teachers make decisions based upon knowledge, goals, beliefs and orientations. Accordingly, developing all of these factors can help promote decision making in the mathematics classroom.

For many years, I have been seeking creative ideas that will enable in-service and pre-service teachers to predict scenarios and unexpected situations in the mathematics classroom. Thus, they will be able to practice mathematical discourse before coming to class and to learn to provide argumentative responses that are quick, accessible and flexible. Teachers' responses in class and their responsibility in developing mathematical Concept development, discussions and discourse have been the topics of much investigation (e.g., [18], [22]). The literature has placed less emphasis on examining training methods for developing discourse management for predicting unexpected classroom situations in advance, particularly training all that through writing.

[26] Describe a fictional dialogue on infinitude of primes between Euclid and Dirichlet and use this as a research method. The current study focuses on pre-service and in-service math teachers who write "fictional dialogues" as part of their training. The goal of this writing is to develop their ability to explain, respond and engage in argumentative mathematical discourse in a learning situation characterized by unexpected situations. The results of the current study indicate that the task of writing fictional dialogues has several advantages. One advantage relates to professional development and renewal. Veteran teachers tend to feel less challenged and less interested in preparing lessons in advance. Writing fictional dialogues challenges them to formulate unexpected mathematical situations for mathematical topics and ideas that for them are seemingly simple and trivial. In writing fictional dialogues, they discovered both mathematical and didactic innovations. Another advantage applies to training. In writing the dialogues, beginning teachers learned to develop written mathematical discourse that explains the essence of mathematical terms. Further, they learned to use visual or other representations in context and practiced giving explanations to learners with a variety of learning styles. Another positive aspect deriving from the research results is related to independent or group writing, with independent writing emerging from recurring attempts at group writing.

II. THEORETICAL BACKGROUND

The current study emerges from the assumption that formulating a program to train pre-service or in-service teachers to enhance mathematical discourse is complex, particularly in unexpected situations in the mathematics classroom. Unexpected situations differ from teacher to teacher due to differences in the extent and depth of their mathematical knowledge, their ability to identify such situations and their ability to make decisions in real time about the didactic concepts appropriate for each situation. Hence, I examined the research literature on major topics related to the current research. These include training pre-service and in-service math teachers by means of writing, the role of the teacher in discourse development and management in the mathematics classroom, mathematical argumentation as a teaching tool and interaction in the mathematics classroom. The conclusions of these studies led me to formulate ideas for a unique intervention "training" program with the potential to promote mathematical discourse in the classroom in general and argumentative mathematical discourse in unexpected situations in particular. In the following sections, I review the relevant literature in these fields and explain how these studies relate to the current research.

Professional Development and Learning through Writing

Teacher training usually incorporates writing through writing assignments about ideas learned in class or as reflection on learning [15], [16], [11]. Turning writing into a goal in and of itself is an innovation in the training of mathematics teachers. Therefore, in order to construct an intervention program that emphasizes writing, I surveyed and studied research that examines the advantages of writing in teaching math and of pedagogy based on writing in general.

In the study by [3], the students learned by writing diaries on mathematical argumentation. The research indicates that the process of writing develops students' in-depth thinking about mathematical concepts as well as underlining erroneous or other perceptions of concepts or phenomena. The writing process and the accompanying feedback prompted the students to write more precisely about mathematics, directed them to give arguments, explanations and reasoning in their writing and taught them to edit and rethink mathematical ideas. From this study among students, I decided to try to generalize the method for adults and to examine the results. Adults with a common professional interest often write together in a process that advances their shared understanding and learning in the field [12].

Griffin & Beatty [8] examined the attributes of shared writing among adults with a common professional interest. Their research pointed to several advantages, including professional and personal growth among the writers, a greater degree of creativity, the generation of new ideas and understandings, diversification in areas of specialization, increased documentation and output abilities, and shared knowledge. Shared writing generates a unified voice, increases feelings of satisfaction and pride in integrating the personal voice into the voice of the group and expresses respect for individual knowledge. Therefore, in this study the writing took place in pairs or in small groups as part of the process of developing skills in argumentative mathematical writing.

IMSCI model was proposed for supporting the writing process, with writing serving as a pedagogical tool for assimilating learning. In the IMSCI acronym, "I" stands for inquiry, "M" for modeling, "S" for shared writing, "C" for collaborative writing and "I" for independent writing. This scaffolding model was integrated into the intervention process in the current study [17].

Spoken or Written Mathematical Discourse

According to Sfard [20], discourse has four characteristics: vocabulary, visual mediators, unique routines and customary utterances. Classroom discourse comprises more than the words spoken. In keeping with the communicative approach, [21] analyzes the learning situation by means of what is said, done, heard and seen in the learning environment. The roots of this approach can be found in participationism theories, discursive psychology and sociocultural theories that conceptualize learning as the process of transforming the learner into a participant in a certain type of activity. In the communicative approach, thinking constitutes an individual's discourse with the self. Such a discourse can yield ideas that express the thinking of those participating in the discourse.

In contrast to those who talk, some people express themselves through writing and symbolic mathematical representations and have difficulty expressing their ideas verbally. Such individuals may eventually become teachers whose skills in developing and conducting mathematical discourse are not sufficiently developed. In most cases, this does not point to a lack of mathematical knowledge but rather to the difficulty teachers experience in translating this knowledge, which perhaps is represented in their minds through nonverbal symbols, into verbal tools. Mathematics teachers must generate significant discourse in their classrooms. Such discourse constitutes an organized and connected collection of all their students' and their own intellectual ideas. The job of the teacher is to conduct a discourse that reflects ideas and encourages participants to discuss these ideas, to endorse or refute them and to arrive at valid and agreed-upon mathematical rules that can be implemented in new situations that are similar or different [28]. How can we promote and cultivate teachers who have the awareness and skills to cultivate this type of classroom reality?

Wagganer [25] Proposed five strategies for supporting meaningful math talk in class. First, teachers must talk with their students and arrive at common insights regarding the importance of math talk in the classroom. Second, teachers are responsible for teaching their students to listen and respond appropriately to one another. Third, teachers must teach their students to write sentence stems to emphasize their responses. Fourth, teachers must teach and demonstrate the difference between explaining and justifying what someone else says. Finally, teachers must provide examples of all these actions in class. The current study implemented all of Wagganer's ideas with pre-service and in-service teachers in the general context of group mathematical discourse and the particular context of written mathematical discourse in unexpected situations in the mathematics classroom.

III. Research Method and Intervention Design

The objective of the current study was to promote the development of spoken and written mathematical discourse among pre-service and in-service math teachers in the context of classroom scenarios they considered unexpected and complex. The training was directed toward developing argumentative mathematical discourse skills through writing, with emphasis on writing fictional dialogues. The research participants included undergraduate students taking a course that taught didactic and pedagogic skills for teaching math in elementary and junior high school and graduate students in mathematics education who teach math to all ages and at all levels, including at the tertiary level. The two groups together totaled 35 students, as half of them were teachers were in fact teachers.

The course comprised several stages. First, the students read the article by [26], [27], [28] about fictional dialogues in order to understand and define fictional dialogues in the context of their unique methodological role in the original article. Next, we adopted the skill of writing fictional dialogues as a tool for developing spoken and written mathematical discourse in lesson planning for unexpected situations in the math classroom. We embraced the following quote with the understanding that we as students

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also seek interesting learning methods. "People are eager for stories. Not dissertations. Not lectures. Not informative essays for stories" (Haven, 2007, p. 8 on [26]).

Third, we defined and formulated conditions determining whether a potential fictional dialogue met the objective. In this stage, we read mathematical dialogues from various sources that resembled fictional mathematical dialogues and we reworked their mathematical discourse so it matched our definition of a fictional dialogue. Fourth, the students independently wrote fictional mathematical dialogues. In the fifth and final stage, the students showed their dialogues to their classmates. This generated an evaluative argumentative discussion and, if necessary, led to redesigning the dialogues. Throughout the course, we documented the sessions and their outcomes.

IV. Definition of "Fictional Dialogue" in the Current Study

The definition of fictional dialogue emerged from agreement among all course participants and included the following characteristics. The dialogue must take place between two people with some sort of major gap between them. This gap may be rooted in culture, age, expertise, historical period (e.g., one speaker lives in contemporary times and the other lived 700 years ago), mathematical knowledge and more. One speaker is an expert in the field and should be able to bridge the gap through argumentative dialogue that leads the two speakers to understanding, definition and agreement on the mathematical topic they are discussing. The expert presents the mathematical explanation using formal intra-mathematical tools and extra-mathematical or other simple, practical and concrete examples and explanations. The non-expert participant's dialogue develops in unexpected directions, so that this participant can surprise the expert with questions or examples that seemingly contradict the mathematical concept under discussion or that present a challenge to the clear, simple and popular explanation. In the dialogue, the two participants express their perceptions of the mathematical topic being discussed, and each attempts to enrich the other's world through the mathematical knowledge at his or her disposal. Through the dialogue, the gap between the speakers becomes smaller in that all the relevant mathematical nuances in the field find expression in the dialogue.

V. Findings

The findings from each stage in the course intervention process were analyzed. Typical categories were found for each stage. Moreover, some categories recurred in all the stages. Figure 1 depicts the findings, including the categories that recurred in all the stages, by means of a linear or a cyclical model.

In this paper, I describe two mathematical events representing two stages of the intervention period. Because the research focuses on the final product — writing — I give two examples of writing and discuss the processes involved in creating them. The two examples, shows that writing fictional mathematical dialogues are training and professional advancement tool for pre-service and in-service math teachers.

The first finding refers to the third stage of the intervention period, in which we redesigned a dialogue and rewrote it as a group fictional dialogue. At this stage, each student individually redesigned the dialogue by writing a new dialogue based on the existing dialogue and thus creating a new personal product that conformed to the required conditions. In the next stage in the joint group work, the students showed their dialogues to their classmates for evaluation, leading to writing an agreed-upon group product. The dialogue is the unified product after the group discussed their differences and went through the entire learning process.

The second finding is presented in the form of an essay written by a student teacher pursuing a master's degree in math education. His essay is the result of independent writing for the course's final assignment.

Table I: Independent vs. Group Writing

Independent writing process			*Spoken and Written Argumentative Discourses*	*Team writing process*			*Spoken and Written Argumentative Discourses*
Progression				*Progression*			
Situation	*Trigger*	*Outcome and activities*		*Situation*	*Trigger*	*Outcome and activities*	
Individual research	*Learning*	*Expanding and deepening personal knowledge of mathematics content*		*Individual and group research*	*Learning*	*Renewed expansion and deepening of individual and group mathematical knowledge content via evidence, data, justifications and authorizations*	
Individual re-learning	*Knowledge Completion*	*Challenge & innovation: Demonstration of ideas as a result expanding and deepening personal didactic mathematical knowledge*		*Team Agreement*	*Knowledge Completion*	*Backup, support and demonstration of old and new individual and group mathematical ideas*	
Regression				*Regression*			
Sharing ideas and feedback	*Misunderstanding*	*Accuracy individual learning*		*Disagreement*	*Misunderstanding*	*Demo of contradictory mathematical ideas*	

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Progression				*Progression*			
Writing	*Clarification*	*Creating an integrated personal idea*		*Sharing ideas and feedback*	*Misunderstanding*	*Restructuring ideas*	
Independent writing				*Co-writing*	*Clarification*	*Creating an integrated group idea*	

These two outcomes point to development of the participants' mathematical and didactic knowledge, development of written mathematical discourse, development of an argumentative process while conducting the dialogue and experience in predicting and managing unexpected mathematical situations at an advanced stage of preparing a dialogue to use in a mathematics lesson.

Group design of a given Dialogue and its Transformation into a Fictional Dialogue

The given dialogue is from an Abbott and Costello movie titled *Buck Privates*:

Abbott: You're 40 years old, and you're in love with a little girl, say 10 years old. You're four times as old as that girl. You couldn't marry that girl, could you?

Costello: No.

Abbott: So you wait 5 years. Now the little girl is 15, and you're 45. You're only three times as old as that girl. So you wait 15 years more. Now the little girl is 30, and you're 60. You're only twice as old as that little girl.

Costello: She's catching up?

Abbott: Here's the question. How long do you have to wait before you and that little girl are the same age?

Costello: What kind of question is that? That's ridiculous. If I keep waiting for that girl, she'll pass me up. She'll wind up older than I am. Then she'll have to wait for me!

In order to determine whether this qualifies as a fictional dialogue, we mapped it to see whether it fulfills the conditions for fictional dialogues formulated in the second stage of the course.

Table II. Mapping conditions for qualifying as a fictional dialogue

	Condition	Fulfills	
		Yes	No
1.	*Dialogue takes place between two people with some sort of major gap between them.*	√	
2.	*One of the speakers, an expert in the field, should be able to bridge the gap.*		X
3.	*The expert provides the mathematical explanation using formal intra-mathematical tools and extra-mathematical or other simple, practical and concrete examples and explanations.*		X
4.	*The non-expert participant's dialogue develops in unexpected directions, as does that of the expert.*	*"If I keep waiting for that girl, she'll pass me up. She'll wind up older than I am. Then she'll have to wait for me!"*	
5.	*The two participants express their perceptions of the mathematical topic as they develop during the dialogue.*		X
6.	*Through the dialogue, the gap between the speakers becomes smaller in that all the relevant mathematical nuances in the field find expression in the dialogue.*		X

The mapping results indicate that the dialogue does not meet the conditions to qualify as a fictional dialogue. Hence, we redesigned the dialogue to fulfill the necessary conditions. Each course participant individually designed and wrote a fictional dialogue. In the next stage, the students as a group combined these individual dialogues into a fictional group dialogue. The group dialogue features an expert "player" called Achilles, provides intra - and extra - mathematical explanations, stresses the perceptions of each of the speakers so that it is clear who represents the erroneous perception and who represents the appropriate perception and stresses the unexpected situation. Using the ideas from the individual dialogues, the group wrote an argumentative fictional dialogue that gap the discrepancy between the speakers to the point of generating an unexpected situation in which the speakers "reverse" their roles, so that the rookie, Costello, triumphs over the expert, Achilles.

The Age Difference Problem: Achilles the Mathematician vs. Costello the Comedian.

Achilles: You're 40 years old, and you're in love with a little girl, say 10 years old. You're four times as old as that girl. You couldn't marry that girl, could you?

Costello: No.

Achilles: So you wait 5 years. Now the little girl is 15, and you're 45. You're only three times as old as that girl. So you wait 15 years more. Now the little girl is 30, and you're 60. You're only twice as old as that little girl.

Costello: She's catching up?

Achilles: Here's the question. How long do you have to wait before you and that little girl are the same age?

Costello: What kind of question is that? That's ridiculous. If I keep waiting for that girl, she'll pass me up. She'll wind up older than I am. Then she'll have to wait for me!

Achilles: Hold on. Let's explain this again. Are you ready?

Costello: Yes. I don't want to lose the girl.

Achilles: I, Achilles, run at a speed of 10 meters per second. My friend the turtle runs 1 meter per second. I decide to give the turtle a head start of 100 meters at the beginning of the race.

Costello: Wait a minute. This is a fable, right? So I want to convert it to apply to me. I gave the girl a forty-year head start. Wow, that's a lot. I am four times older than she is! And you run ten times faster than the turtle. Great, I get it.

Achilles: So let's go back to the girl. You are 40 years old and the girl is 10 years old.

Costello: I think I am four times older than she is. Let's create a situation in which I'm ten years old and she's one year old, and then I'll have an easier time understanding. Like with the turtle . . .

Achilles: Unnecessary. We'll stay with two stories with different ratios and we'll still create a situation where you can understand the problem.

Costello: I understand part of it. When I am 45, she'll be 15 and then I'll be three times older than she is.

Achilles: Good. So I let the turtle run 100 meters ahead of me. How much time does it take the turtle to run 100 meters?

Costello: 100 meters – 100 seconds. So let's assume the girl is one year old and I'm 100 years old . . .

Achilles: Not necessary. After 100 meters, or 100 seconds, I begin running.

Costello: That's complicated. You'll run 100 meters in 10 seconds and catch up with the turtle.

Achilles: Yes, but while I'm running 100 meters the turtle is continuing to run . . . so I can't catch up with him.

Costello: Wow, that's really complicated. The turtle doesn't stop and I also am getting older and so is the girl . . . what's the outcome?

Achilles: It's not so complicated . . . Let's continue with the girl. When you are 60, the girl will be 30, so you'll be twice her age. From there on, the ratio between your ages is not a whole number. For example, when you are 80 and she is 50, the ratio between your ages will be 50:80, or five eights.

Costello: Hold on a minute. I need to calculate these ratios myself. It's easier for me to understand the ratio when I'm 90 and she's 60 because then I'll be less than twice her age. My age divided by her age is one and a half, so the ratio between us is less than one and a half. When I'm 120 and she's 90, the ratio between us will be 3:4. Her age will be three-fourths of my age. So I'll probably die before I'm able to marry the girl I love.

Achilles: Yup, it seems that you won't get married. But she's not going to pass you up like you originally claimed.

Costello: She won't pass me up only because I'll be dead. If I live to 150, she will pass me up.

Achilles: And how is that?

Costello: When I'm 150 she'll be . . . 120. Oh no. Now she'll die. She'll be dead!

Achilles: She won't die. She'll continue living. But she'll never pass you up.

Costello: So by this calculation she'll be . . . ah . . . If 150 is five times 30 years, 120 is four times 30 years. So her age is four-fifths of my age . . . she'll be four-fifths of my age!

Achilles: Do you get it now? She will never pass you up. As you get older, the ratio between your ages gets smaller but does not disappear.

Costello: So let's assume I'm 240 years old. How old will she be??? She will always be 30 years younger than me, so she'll be 210 years old. So her age will be seven-eighths of my age. It appears we are slowly advancing to the point where we're the same age.

Achilles: No. That's not right. Let's go back to my turtle.

Costello: I'm not going back to your turtle because I've discovered the problem and also the solution. The girl and I will never get married because there will always be a fixed difference of 30 years in our ages. But there is not a fixed difference of 100 meters between you and the turtle because around 12 seconds after the beginning of the race you will already catch up with the turtle. In ten seconds you run 100 meters and in another two seconds you run another ten meters, so the race is over because the turtle continues to trail behind you.

Achilles: Does that mean that the age difference problem is not representative of infinity.

Costello: Now we've switched roles. I'm the mathematician. What are you? The concept of infinity can be represented if the beautiful girl and I live forever and do not die. The ratio of our thirty-year age discrepancy changes. From a situation in which I'm four times her age and then three times her age, we get to a situation where the ratio is 7:8, and we can go on to 8:9 and even further. While the difference in our ages is still thirty years, with time the relative difference in our ages gets smaller. In contrast, when you race against the turtle, a quick calculation tells me that you'll overtake your opponent after 12 seconds. The 100-meter difference between you is not fixed because you "grow at different rates." That is, you each run at a different speed and you are ten times faster than the turtle.

Achilles: So the example I gave you to explain the girl's age is not suitable.

Costello: I love the girl so much that I managed to solve the problem. Your example is not suitable because you and the turtle run at different rates. If the girl was described as getting older at a different rate, if each year she became five years older . . . then I think we could get married.

Achilles: Wow, you've really discovered the difference between the problems. They are not at all similar.

Costello: Nevertheless, tell me what we can learn from the turtle example?

Achilles: Maybe to calculate equations of position as a function of time.

Costello: That's really complicated. The turtle helped me understand the decreasing ratio in the difference between the girl's age and mine. The concept of speed is the ratio between distance and time, and when I understood this as a ratio, I understood the age difference problem.

Achilles: Unbelievable. You solved both the age difference problem and Zeno's Dichotomy Paradox, and you also determined what is similar and what is different in these two problems.

Costello: Despite everything, maybe you can still introduce me to the girl.

Dialogue analysis "Age Difference Problem"

In the above dialogue, the students completed all the conditions that were missing from the original given dialogue. They created two fictional characters and delineated a significant historical and mathematical gap between them. They defined an expert speaker who led the dialogue. They formulated intra - mathematical explanations (e.g., speed as the ratio between distance time) and extra - mathematical explanations (e.g., representing the concept of infinity by means of the girl and Costello, who grow forever and never die) for the age problem and for the paradox of Achilles and the tortoise. Furthermore, they created two unexpected situations in the dialogue. One was the comparison between the age problem and the Achilles paradox. The other was that Costello understood the difference between the problems and claimed that the turtle problem differs from the age problem ("Now we've switched roles. I'm the mathematician. What are you?"). They created a specific explanation for the problem and its concepts and accurately differentiated between the two problems.

Using the dialogue, they understood that the age problem demonstrates Costello's misconception about the age gap, as he thought the gap would decrease over time.

In contrast, the turtle paradox shows that the gap between the turtle and Achilles is not fixed and that the distance decreases with time. Using numbers, the students demonstrated the two situations, showing that the gap in the age problem remains constant while the distance between the turtle and Achilles continues to diminish. At this stage, they reduced the gap between the speakers' dialogue.

During the group formulation, the students explored ide--as and mathematical explanations. They designed and formulated the dialogue as a group exercise, so that in cases of disagreement they stopped and sought a consensus in the group.

Based upon my documentation and the students' testimony, the group writing experience enabled them to observe the situation in a variety of ways. It gave them the opportunity to understand the perceptions of both Costello and Achilles from various perspectives.

In the students' opinion, the group exercise in class was a safer place to express themselves and create ideas than in the classroom with students. They claimed that the experience they gained in reshaping dialogues and making them fictional promoted their sense of expertise in managing mathematical discourse.

Individual Design and Formulation of Fictional Dialogues

The second finding refers to the advantages deriving from independent individual experience in writing a fictional dialogue. The following fictional dialogue was written by a master's degree student. The dialogue deals with an ostensibly trivial topic that all high school math teachers must deal with: how to transform an ellipse into a circle. In the dialogue, the astrophysicist Johannes Kepler (1600) talks with a fictional character named GeoGebry who plays the role of the man who programmed the GeoGebra software package (2016). The student was faced with the challenge of finding an unexpected situation in teaching simple and trivial math. He then had to write a dialogue conveying new mathematical and didactic ideas for this situation.

VI. From Ellipse to Circle

Kepler: Hi, GeoGebry. I've been told that while I've been resting in the world to come, you've been developing a new computing tool to represent mathematical relations between shapes and bodies on planes and in space. I must confess this would have helped me formulate my laws several years earlier.

GeoGebry: Yes, the GeoGebra software tool helps in making generalizations.

Kepler: My friend Brahe spent twenty years sketching the relations between the planets. When I got my hands on his sketches, I formulated the first law of planetary motion.

GeoGebry: Yes, we are promoting a software package aimed at representing all mathematical relations between functions, objects or other mathematical relations and graphs. I dare you to challenge me with a new development idea.

Kepler: In my time, I introduced the innovative idea of the movement of the planets around the Earth.

GeoGebry: To the best of my recollection, you formulated several orbits of planets revolving around the Earth and the sun. Give me the figures and I will prepare software that represents the objects given their size, the distance between them, angles, etc.

Kepler: I will be precise. Let's begin with the formulation of my first law: *The orbit of every planet is an ellipse with the Sun at one of the two foci.*

GeoGebry: I'll show you a drawing that provides a graphic demonstration of the law. See if it represents the law as you formulated it. Here is the drawing[1]:

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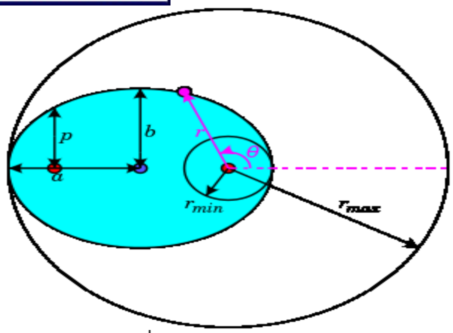
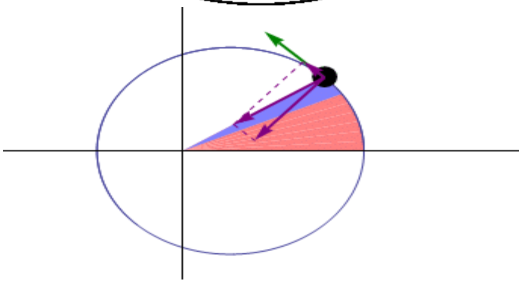




Kepler: Back then, I also made similar sketches. I think my notebooks went to Newton.

GeoGebry: It's not the same sketch. This is a single screen shot. In my software, I can move all the points: the sun, the planets, the second focal point. I can change the size of the primary and secondary axes, and as I move them I can see the path and can calculate whatever you want – the angles, the sides of the triangles, the distance from . . .

Kepler: I'm not sure you understand the precise innovations I made in this field.

GeoGebry: I focus on technology that can dynamically represent the relations.

Kepler: The change you've introduced focuses mainly on the transition from the old geocentric model to the heliocentric model that began to take hold. The geocentric model claimed that Earth is the center of the universe and that the Sun and the planets revolve around it. This model was undermined by my work and that of Galileo Galilei, who worked at the same time in Italy. According to the heliocentric model, the Sun is at the center of the universe, with Earth and the other planets revolving around it.

GeoGebry: In other words, until you formulated your law, the planets moved along a circular orbit.

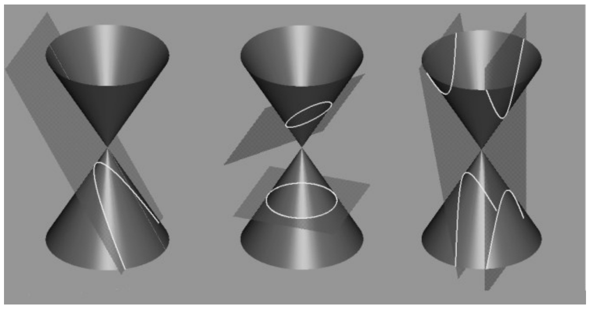
Kepler: Right. My innovation was my claim that planetary motion is elliptical. And by the way, a circle is a special case of an ellipse.

GeoGebry: That's interesting. I thought an ellipse was a special case of a circle. Just a minute. What difference does it make which is the special case and which is the general rule?

Kepler: Okay, here's your first challenge. Use your tools to determine which is a special case of the other.

GeoGebry: Both an ellipse and a circle are conic sections that I can represent using different graphical

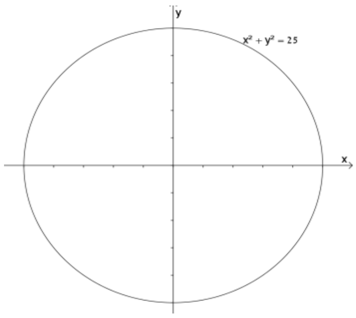
GeoGebry: Both an ellipse and a circle are conic sections that I can represent using different graphical means[1] :



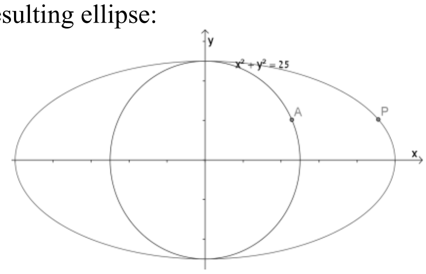
This picture also shows hyperbolas and parabolas…

Kepler: Conic sections do not necessarily explain the relations between the shapes as general and special cases. Think about the special case. Let's continue

GeoGebry: But I can dynamically draw a circle and change its shape to an ellipse. Here is a circle where $x^2 + y^2 = 25$.



Now I'll stretch it along the X-axis to two times its size. For every point (x,y) on the circle there is a new corresponding point (2x,y). Here is the resulting ellipse:



[1] Source: en.wikipedia.org, CC BY-SA 3.0, https:// commons. wikimedia.org/w/index.php?curid=4210181

And here is the equation representing the ellipse:

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

Kepler: Okay, you're getting close. If you understand that an ellipse and a circle describe the orbits of the planets, you can more easily understand the nature of the relations between the shapes. The extent of the ellipse's "flatness" is called eccentricity. In your drawings, the circle turns into an ellipse. In my sketches as well, the ellipses become more and more eccentric from left to right. A circle is considered a special case of an ellipse with zero eccentricity. As the ellipses become flatter, their eccentricity approaches the value of 1. Therefore, the eccentricity of any ellipse ranges from 0 to 1.

GeoGebry: I still don't understand why the circle is a special case.

Kepler: As I said at the outset, what we know about ellipses we learned from planetary movement. The movement of the planets is elliptical, but the eccentricity[2] of the orbits is so small for most planets that at first the orbits appear to be circular. For most of the planets, we need to measure the geometry precisely in order to determine that they are not moving along circular orbits but rather elliptical orbits with very small eccentricity. This is something that could have been noticed using GeoGebra because the software depicts reality at a high level of precision. Moreover, in addition to the sketches, the software provides numerical data to represent the situation. These numbers would help show that the shape approaches a circle.

GeoGebry: I'm beginning to understand where you're going.

Kepler: Pluto and Mercury are exceptions. Their orbits around the Sun are very eccentric, and the graphs describing them clearly show their non-circular motion.

GeoGebry: According to your first law, Kepler, the Earth rotates around the Sun along an elliptical orbit. The Earth's orbit is almost circular. For you to arrive at the conclusion you claim our software could have easily depicted, you and Newton needed to use additional laws of gravity.

Kepler: The Earth's eccentricity is only 0.0167!! Pluto, for example, has an orbit that is not at all circular, with eccentricity of 0.2488. Also note that the Sun is not exactly at the center of the elliptical orbits of the planets revolving around it.

GeoGebry: An ellipse has a point that is a bit far from the center of the ellipse, known as the focus. The Sun is located at the focus of the ellipse. I showed this in my first sketch of the sample orbit.

Kepler: Because the Sun is at a focal point and not at the center of the ellipse, each planet moves closer to and further away from the sun during each revolution. Because most of the movements are elliptical, circular orbits are the exceptions. Hence, a circle is a special case of an ellipse.

GeoGebry: That's amazing! I've really broadened my understanding of the relations between a circle and an ellipse. Now I will explain to my students the relationship between the general equations for ellipses and circles and I may also construct the relationship to the equation for hyperbolas.

Dialogue Analysis "From Circle to Ellipse"

The writer is a high school math teacher who decided to write a trivial mathematical exercise, believing and hoping the writing would lead him to a mathematical or didactic innovation. At first, he wrote a trivial dialogue without a gap between the two speakers and was therefore not satisfied with the results. Then he decided to read various sources discussing the relations between a circle and an ellipse to enrich his knowledge and thus produce an unexpected situation. While reading about Kepler's laws of planetary motion, the statement that "a circle is a special case of an ellipse" took him by surprise. He felt challenged to figure out this relationship and to find ways to explain and represent it.

He realized that up to then he had understood the statement backwards: "An ellipse is a special case of a circle." Using Kepler's laws of planetary motion, he was able to formulate intra - mathematical explanations for the relationship between an ellipse and a circle. According to the teacher and as we can see from the dialogue, the dialogue moves from abstract reality to a reality that can be represented using a sketch or a mathematical explanation, thus highlighting the gap between the two speakers. Kepler refers to an intra - mathematical explanation that describes reality that is far away and must be imagined — planetary movement. GeoGebry tries to use drawings to represent and understand reality. Kepler's mathematical advantage over GeoGebry is clear, and despite the time difference between them, the expert Kepler narrowed the mathematical gap.

The teacher's goal was to show that using software without understanding concepts and critical features of and relationships between mathematical shapes would not necessarily produce learning or meaningful mathematical representation. The student wrote a dialogue that explains the relationship between the equation of a circle and the equation of an ellipse. He then connected this explanation to the movement of the planets and their positioning along the main and secondary foci.

Writing an independent fictional dialogue is a complex task. The interaction is with mathematical content that the writer must learn independently rather than with other learners who can contribute their knowledge, explanations and illustrations. After their group experience and based

[2] Eccentricity is an estimation of the flatness of a conic section, usually signified by the letter e. Eccentricity is dependent upon the type of conic section. The flatter the object, the larger its eccentricity.

on examples of successful fictional dialogues, the students acquired the tools to produce the five necessary conditions for a fictional dialogue. Determining these five conditions enabled them to examine themselves at any stage of the writing to determine whether they did indeed meet the conditions.

VII. Summary of Findings

In summarizing the section on findings, I quote from one student's reflections in the final assignment. "To write a mathematical dialogue is to contradict mathematics. Math is writing that is symbolic, concise and subjective. Writing a dialogue is totally contrary to everything I ever thought about mathematical writing. The dialogue included the participants' subjectivity, not my own. It included the efficiency of the explanation that the speaker required and not necessarily mathematical conciseness as I perceive it. It contained many words and sentences, because there is no other way to bridge the gap between the speakers. It exhausted me as the writer, but it also made the challenge of teaching accessible to me. I was forced to think of a variety of possible scenarios and to manage these as a teacher. I benefited by learning new math and acquiring new tools presented by the group. I became familiar with a tool for developing my skills as a teacher" (Yoav, 2016).

VIII. Discussion and Conclusions

The quotation at the end of the findings section describes the tension between writing and math. Even after Yoav began believing in writing, he had to generate writing that differed from his beliefs and organize his own thoughts and ideas in accordance with the situation. The writing led him to define new scenarios in the mathematics classroom and to find didactic solutions to manage these scenarios. To date no studies have considered writing in math education as a tool for professional training and development. The current study is a pioneer in this field. The research was inspired by studies that examined student writing in math classrooms [3] and writing-based pedagogies [11]. The study implemented Read's [17] method using the IMSCI model. Implementing this model one step at a time was found to be effective and to validate the results of studies claiming that only theories that are practically applied in the training process can be properly implemented in the field [1], [4] [6]. That is, it would have been more effective to teach the theory of fictional dialogue in the course and then to practice it step by step (IMSCI) through actual writing.

In the following paragraphs, I list the six (rules) conditions for the determining whether a dialogue qualifies for use as a tool and their advantages as conditions for developing teaching skills, as evidenced in the two intervention groups.

The Dialogue takes Place between two people with some Sort of Gap between them.

The advantage of this condition is that it prepares students for mathematical discourse with all types of interlocutors, ranging from stimulating individuals with more math expertise than the student writing the dialogue to novice with scant knowledge of math. This gap is the first opportunity for learning because it is likely to generate situations of intellectual or emotional imbalance [9]. In the age difference problem, the discrepancy between the comedian and Achilles is clear and well defined, with Achilles serving as the mathematical expert. In the example of converting a circle to an ellipse, there appear to be two experts — the math expert Johannes Kepler and the software expert GeoGebry. During the course of the dialogue, the gap widens to Kepler's advantage.

One of the speakers, an expert in the field, should be able to bridge the gap. This condition obligates the writer to define the expert and to write the dialogue based upon the expert's personality. To be competent writers, students must know and read and study the relevant literature (Shulman, 1987). While preparing to write, students develop mathematical as well as didactic expertise in the field. [2] Speaks of three different modalities: one-to-one interaction, peer, and computer/ penpaper - based scaffolding.

At the beginning of the dialogue, the discrepancy between Achilles and Costello widens precisely because of Achilles' example, though it later diminishes. The gap between Kepler and GeoGebry becomes smaller only toward the end of the dialogue when the intra-mathematical explanations clarify that the circle is a special case of an ellipse.

The expert provides the mathematical explanation using formal intra-mathematical tools and extra-mathematical or other simple, practical and concrete examples and explanations. This condition obligates the writer to think like an expert and to formulate intra- and extra-mathematical ideas in order to convey his or her ideas, like the case of problem posing [13]. Kepler directs the dialogue toward an explanation of planetary motion that seems to be extra-mathematical. Yet in essence, it explains the intra-mathematical notion of elliptical versus circular motion as well as the relations between the shapes and the degree of flatness.

The non-expert participant's dialogue develops in unexpected directions. To fulfill this condition, the writer of the dialogue must think of an unexpected scenario at each stage of the writing. This condition can drive develop of teachers' mathematical and didactic competencies [14].

"We agree with Ellerton (2013) when she says: "For too long, successful problem solving has been lauded as the goal; the time has come for problem posing to be given a prominent but natural place in mathematics curricula and classrooms" (pp. 100 –101) and our research shares this idea [14] " this research is also share this idea, and believe that writing fictional dialogue is like writing problem posing.

For example, the notion that a circle is a special case of an ellipse was unexpected, as was the discovery that the problem of Achilles and the turtle was not parallel to the age difference problem because of the issue of a fixed difference versus a variable difference.

The two participants express their perceptions of the mathematical topic being discussed. To meet this condition, the writer of the dialogue must recognize and understand a variety of approaches in the field [7] and must maintain balance between the two speakers. Each believes his ideas are rational and coherent until confronted with another idea that changes the direction of his thinking. Costello coherently expresses the notion that the girl will grow up and pass him up, and Kepler rationally conveys the notion that the relations between a circle and an ellipse resemble planetary motion.

*Through the dialogue, the gap between the speakers becomes smaller in that all the relevant mathematical nuances in the field find expression in the dialogue***.** This condition is essential for the dialogue to reach an optimal conclusion. Understanding and relating to the mathematical nuances is critical [23]. In class, teachers can sometimes generate agreement based upon their authority. In a dialogue, it is not possible to generate this type of agreement. Rather, the writer must make sure that the speakers do indeed arrive at mutual understanding of the topic under discussion. The two sample dialogues both end after all the questions, considerations and surprises have been settled.

"Education policy should aim to promote instructional methods that are easy for teachers to implement and have demonstrable, positive impact on student learning" [24], the current study applies that.

The current study includes several innovations. One is its definition and formulation of rules to determine whether a "fictional dialogue" qualifies as a training tool to promote spoken or written argumentative mathematical discourse. Another is its "validation" as a tool that achieves its objective and challenges veteran teachers as well to implement it in promoting and developing teachers' mathematical discourse to manage unexpected situations in the math classroom. A third innovation is that the study considers writing to be a tool for organizing knowledge and thoughts that encourages individual and group interaction in mathematical argumentation.

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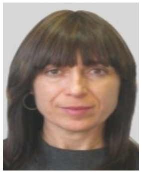

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